Rare events and breakdown of simple scaling in the Abelian sandpile model

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Due to intermittency and conservation, the Abelian sandpile in two dimensions obeys multifractal, rather than finite size scaling. In the thermodynamic limit, a vanishingly small fraction of large avalanches dominates the statistics and a constant gap scaling is recovered in higher moments of the toppling distribution. Thus, rare events shape most of the scaling pattern and preserve a meaning for effective exponents, which can be determined on the basis of numerical and exact results. [S1063-651X(98)50209-7]

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The sandpile model was introduced as a first example of self-organized criticality (SOC) [1]. SOC can manifest itself in the stationary state of slowly driven, dissipative systems: intermittent activity bursts occur then over all allowed scales, implying power law correlations in both space and time.

The analytic tractability of the Abelian sandpile (ASM) has allowed a number of exact results, especially in two dimensions [2–4]. This considerably enhanced the theoretical interest of this prototype model. On the other hand, in spite of some remarkable advances [2,3,5,6], a satisfactory understanding of ASM scaling is still missing. Unlike many stationary state properties, the exponents describing avalanche size distributions are not known exactly, yet, and remain a major challenge in the whole field of nonequilibrium critical dynamics.

By analyzing the avalanches of a two-dimensional (2D) sandpile in terms of waves [6], Priezzhev *et al.* [7] proposed a scaling picture which led to conjecture all exponents, including those describing the avalanche distributions in terms of covered area and toppling number. Unfortunately, determinations of such exponents are notoriously difficult and there is no compelling numerical support of the conjecture [8,9]. A more important objection came from an extensive study [10], which showed the illegitimacy of a basic assumption made in Ref. [7], concerning the relative sizes and scaling of successive waves.

In this Rapid Communication we reconsider scaling in the 2D ASM. We show that intermittency and conservation lead to a breakdown of finite size scaling (FSS) in the distribution of topplings. This distribution has manifest multiscaling properties. However, due to a very peculiar role played by a class of rare, large avalanches, a standard FSS description can be effectively recovered, if one focuses on the higher moments of the distribution. Quite remarkably, within our picture the effective scaling exponents of the toppling number distribution are determined by an asymptotically vanishing fraction of all avalanches and take the same values conjectured in Ref. [7]. Beyond these results, our analysis exemplifies a path towards the correct characterization of

criticality which should be followed also for other models and phenomena out of equilibrium.

Attempts to identify possible multiscaling features in avalanche distributions started soon after the introduction of SOC [11]. For 1D sandpiles evidence has been most often definitely in favor of multiscaling [11–13]. However, the physical origin of such multiscaling was never fully elucidated. On the other hand, for systems in $d>1$ standard FSS has never been put into serious doubt, so far, and, with almost no exception [11], has always been assumed in both numerical and theoretical work.

We consider a square lattice box of side L. At each site i an integer variable $z_i=1,2,\ldots$ represents the number of grains. If $z_i>z_c=4$ the site topples, i.e., $z_j\rightarrow z_j-\Delta_{ji}$, where Δ is the discrete Laplacian. Thus, when toppling, site i loses four grains, giving one of them to each of its neighbors. At the border, dissipative Dirichlet conditions are assumed, so that one grain per toppling leaves the sandpile. Given a stable configuration ($z_k\leq z_c\forall k$), the dynamics starts by adding a grain at a randomly chosen site l ($z_l\rightarrow z_l+1$). If $z_l+1>z_c$ the site topples, and, if the case, other sites topple, as a consequence of this first instability, until a new stable configuration is reached. The sequence of all such topplings constitutes an avalanche.

Grain addition is a slow driving mechanism compensated by border dissipation, which allows one to reach the stationary SOC state after sufficiently many additions. If one considers a large number of bursts at stationarity, on average, one grain per event must be dissipated, because each avalanche originates from addition of one grain. However, this is still poor information on the dissipation mechanism. Indeed, outflow of grains occurs intermittently, concentrating on a small fraction of avalanches, which are separated by random sequences of nondissipative bursts. This fraction approaches zero for $L\rightarrow\infty$. Thus, dissipating avalanches increase indefinitely with L their outflow. This fact has crucial consequences on the role played by this subset of avalanches on the global statistics.

For the distribution P of the total number of topplings in an avalanche, s, FSS would imply an asymptotic form $P(s,L)\simeq s^{-\tau}p(s/L^D)$, for $s,L\rightarrow\infty$, where D is a capacity fractal dimension of the topplings. In order to better identify the role of rare, dissipating avalanches, we sampled only avalanches generated by grains added right at the center of

the $L \times L$ box. In this way avalanches which dissipate a number of grains $n > 0$ at the borders are always necessarily large, since they span a distance of order L. Such central seed setup is also most suitable for our study of correlation functions. However, as discussed below, our results can be obtained also with random additions over the whole box. Between different central seed additions we decorrelated the system with a sufficiently large number of arbitrary avalanches.

We determined the outflow probability distribution $O(n,L)$ for $L = 32$, 64, 128, 256, and its moments $\Sigma_n O(n,L) n^q = 1/N \Sigma_i n_i^q = \langle n^q \rangle_L$, where N is the number of sampled avalanches ($\leq 10^8$) and n_i is the outflow of the ith one. Also, the inverse first return frequency, i.e., the average number of avalanches T between two successive nonzero outflows was determined. The intermittent character of the stationary state is best revealed by the law $T(L) \propto L^\zeta$, which is satisfied with $\zeta = 0.50 \pm 0.01$. Intermittency and stationarity determine a peculiar scaling of the moments of O. Indeed, the fraction of avalanches with $n > 0$ is of order $L^{-\zeta}$. Thus, $\lim_{q \to 0} \langle n^q \rangle_L \propto L^{-\zeta}$. On the other hand, grain conservation at stationarity also requires $\langle n \rangle_L = 1 = L^0$ (one grain per avalanche added). Thus, by defining in general $\langle n^q \rangle_L \propto L^{-\sigma_n(q)}$, one expects $\sigma_n > 0$ ($\sigma_n < 0$) for $0 < q < 1$ ($q > 1$). We verified numerically that the average outflow, restricted to avalanches with $n > 0$, $\langle n \rangle_{1L} = 1/N_1 \Sigma_{n_i > 0} n_i$ (N_1 is the number of avalanches with $n > 0$), grows as T, i.e., $\propto L^\zeta$. These avalanches rarefy and grow in size for $L \to \infty$. The most simple scaling to expect is $\langle n^q \rangle_L \simeq L^{-\zeta} \langle n^q \rangle_{1L} \propto L^{\zeta(q-1)}$ for $q > 0$. This implies a constant gap $\sigma_n(q) - \sigma_n(q-1) = -\zeta$ for $q > 1$, i.e., a linear behavior for σ_n, consistent with FSS. Such a simple picture is confirmed by our numerical results. The behavior of σ_n is rather close to linear for all $q > 0$ with a gap $\simeq -1/2$ and $\sigma_n(0) \simeq 1/2$. Some previous determinations of ζ in the literature fully agree with the present one [11,14] and we conjecture $\zeta = 1/2$ exactly.

In order to elucidate the role of outflowing avalanches in determining the distribution P of the number of topplings, we define $g_i(\mathbf{r},L)$ as the number of topplings induced at site $\mathbf{r}$ during the ith avalanche. In view of the toppling rules, we clearly have

$$n_i = \sum_{\mathbf{r}} \Delta g_i(\mathbf{r},L), \qquad (1)$$

where the Laplacian acts on $\mathbf{r}$ and the sum extends to the whole box. On the other hand, $\Sigma_{\mathbf{r}} g_i(\mathbf{r},L) = s_i$ is the number of topplings in avalanche i. Thus, dimensional analysis alone would suggest

$$\langle s^q \rangle_L \propto L^{2q} \langle n^q \rangle_L, \qquad (2)$$

i.e., $\sigma_s(q) = -2q + \sigma_n(q)$. However, in spite of the strict linearity of σ_n assumed above, a constant gap behavior for σ_s, as suggested by Eq. (2), is not acceptable. First of all, since $s > 0$ is not selective of outflowing avalanches, we cannot have $\lim_{q \to 0} \langle s^q \rangle_L \propto L^{-\zeta}$. Thus, $\sigma_s(0^+) \neq \sigma_n(0^+)$. Indeed, we know that avalanches with $s > 0$ are a nonzero fraction of the total sampled for $L \to \infty$ [4]. This implies $\lim_{q \to 0} \sigma_s(q) = 0$. In addition, $\langle s \rangle_L = \Sigma_{\mathbf{r}} \langle g(\mathbf{r},L) \rangle_L \propto L^2$ has been rigorously shown on the basis of stationarity and con-
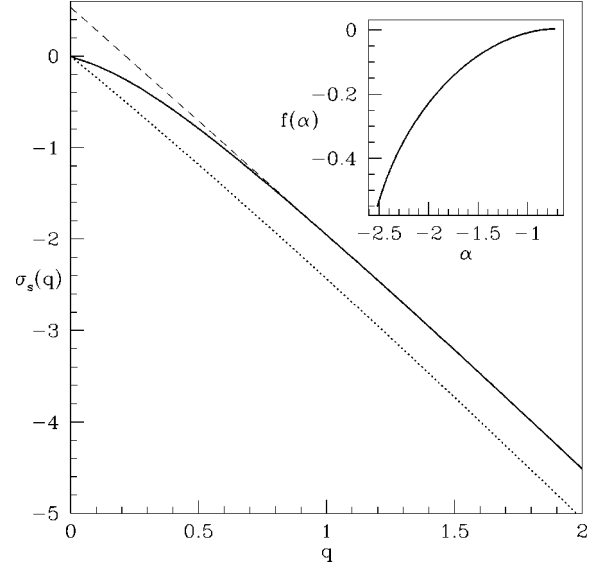

FIG. 1. Plots of σ_s (solid line) and σ_{1s} (dotted line); the dashed line has slope -2.50 ± 0.05 and y intercept $\zeta = 0.50 \pm 0.01$. Inset: plot of $f(\alpha)$ vs α extrapolated from avalanche size distributions. Assuming $f(\alpha(\infty)) = -1/2$ we estimate $\alpha(\infty) = -2.5 \pm 0.1$.

servation [2]. Thus, while Eq. (2) suggests $\sigma_s(q+1) - \sigma_s(q) = -2.5$ for all $q > 0$, basic properties of the nonequilibrium stationary state impose $\sigma_s(1) - \sigma_s(0) = -2$ exactly. It remains to be decided whether a gap -2.5 still applies to at least part of the moments of P. Below we produce strong evidence in support of such a conclusion.

Based on $L = 64$, 128, 256, 512, we extrapolated σ_s as reported in Fig. 1. We find $\lim_{q \to 0} \sigma_s(q) = 0$, and, within an accuracy of 10^{-3}, $\sigma_s(1) = -2$, as expected. Most remarkably, moments of order $q > 1$ appear to conform very accurately to the constant gap -2.5 ± 0.1 expected on the basis of Eq. 2 [e.g., $\sigma_s(2) = 4.5 \pm 0.05$]. Moreover, for $0 < q < 1$, σ_s behaves nonlinearly, clearly confirming multiscaling for P. Outflowing avalanches alone obey simple scaling to very high accuracy. In Fig. 1 we report the moment exponent $\sigma_{1s}(q)$ for the distribution P_1 of $n > 0$ avalanches. Their dominance in the cumulative statistics of all avalanches is such that they impose their constant gap -2.5 as soon as $q > 1$.

An alternative way to discuss multiscaling is through the spectrum of singularity strengths $f(\alpha)$, i.e., the Legendre transform of σ_s [15]: $\alpha = d\sigma_s/dq$, $f(\alpha) = -\sigma_s + q\alpha$. $\alpha(q)$ and f are defined with reference to a saddle point [$s = s^*(q)$] evaluation of $\langle s^q \rangle_L$ in the $L \to \infty$ limit [$s^*(q) \simeq L^{-\alpha}$, $P(s^*,L) \simeq L^{\alpha+f(\alpha)}$]. The inset in Fig. 1 shows the behavior of f as derived by directly transforming our $L = \infty$ extrapolated data for σ_s. Alternatively one can extrapolate ensemble evaluations of f and α at finite L. The discrepancy of such different determinations allowed us to estimate their accuracy. The shape of $f(\alpha)$ is of course consistent with the above results for σ_s: in particular, the fact that $f(-5/2) = -1/2$ is very well satisfied, agrees with the constant gap -2.5 expected for σ_s and with the fact that the linear continuation of the asymptotic σ_s curve (dashed curve in Fig. 1) intercepts the vertical axis at $y = \zeta = 1/2$.

We verified that the above properties of P and σ_s remain substantially unaltered if sampled avalanches are created by
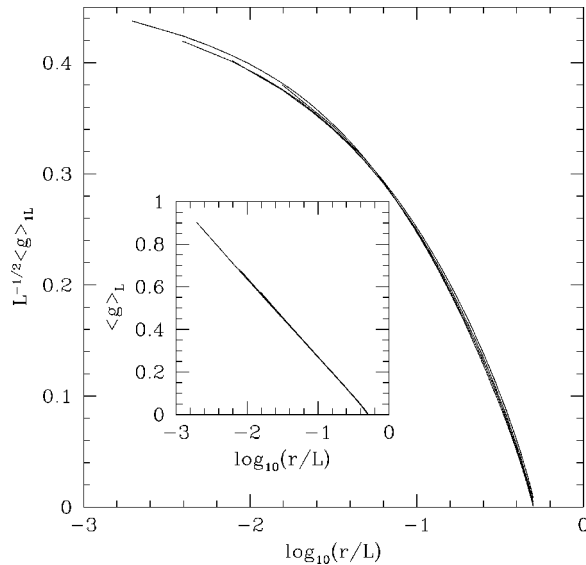
FIG. 2. Collapse fit for $\langle g(\mathbf{r},L)\rangle_{1L}$. The inset reports a similar collapse for $\langle g(\mathbf{r},L)\rangle_{L}$. Data refer to $L=64, 128, 256, 512$.

random grain additions, occurring with uniform probability on the whole box. In particular, $n>0$ is still a condition able to collect the subset of large avalanches dominating for $q>1$ in the $L\to\infty$ limit. Of course, in this case the subset includes also many bursts of activity (e.g., boundary avalanches), whose importance becomes negligible in the thermodynamic limit. The central seed scheme offers the relevant advantage of more rapid convergence with comparable sizes and statistics.

The constant gap for $q>1$ and the behavior of $f(\alpha)$ can be fully understood on the basis of the dominance of rare outflowing avalanches. In our sampling, such avalanches are also the largest, as far as distance spanned is concerned. We can write

$$\left\langle \left(\sum_{\mathbf{r}} g(\mathbf{r},L) \right)^q \right\rangle_L = \frac{N_0}{N} \left\langle \left(\sum_{\mathbf{r}} g(\mathbf{r},L) \right)^q \right\rangle_{0L}$$
$$+ \frac{N_1}{N} \left\langle \left(\sum_{\mathbf{r}} g(\mathbf{r},L) \right)^q \right\rangle_{1L}, \quad (3)$$

where N_0 is the number of not dissipating avalanches, and the subscripts of averages indicate restriction to the corresponding sets. In Eq. (3), of course, $N_1/N\simeq L^{-\zeta}$, and $1-N_0/N\simeq L^{-\zeta}$.

For $q=1$, $\langle g(\mathbf{r},L)\rangle_L$ and $\langle g(\mathbf{r},L)\rangle_{1L}$ in Eq. (3) satisfy very well FSS, but in different forms. Our data for $\langle g(\mathbf{r},L)\rangle_L$ are consistent with a nearly logarithmic dependence on r/L (up to effects due to the upper and lower cutoffs). This is illustrated by the collapse plot in the inset of Fig. 2. Such logarithmic behavior of $\langle g\rangle_L$ agrees with the fact that this function has to coincide with the inverse Laplacian on the box. Indeed, due to the local conservation of grains, which holds on average for the whole sample of avalanches [2], $\langle g\rangle_L$ must satisfy a Poisson equation with a unit source in the origin $\mathbf{r}=\mathbf{0}$. On the other hand, for $\langle g(\mathbf{r},L)\rangle_{1L}$ one cannot invoke local conservation. In fact, as shown by the collapse plot in Fig. 2, our data are very well consistent with a form

$\langle g(\mathbf{r},L)\rangle_{1L}\simeq L^{1/2}g_1(\mathbf{r}/L)$, and the function g_1 is clearly not logarithmic. The peculiar anomalous scaling dimension $1/2$ appearing in $\langle g\rangle_{1L}$ should be identified with ζ. The collapses in Fig. 2 show that $n>0$ avalanches are indeed much richer in topplings than the global average. The average number of waves in each outflowing avalanche, coinciding with $\langle g(\mathbf{0},L)\rangle_{1L}$ [3,6], is proportional to $L^{1/2}$. In fact, if referred to all avalanches, such an average number grows only logarithmically with L [3], as we verified to high accuracy. Both terms on the right-hand side of Eq. (3) give a contribution proportional to L^2 when $L\to\infty$, as expected for the left-hand side.

The singularity strength $f(\alpha)$ is also fully consistent with the properties discussed above. $f(-2.5)=-1/2$ indicates that outflowing avalanches, which are a fraction $L^{-1/2}$ of the total, have a fractal dimension $D=2.5$. In fact, within the multiscaling framework one obtains a whole continuum of fractal dimensions $D_q=[\sigma_s(1)-\sigma_s(q)]/[q-1]$ for avalanches. D_q [15] ranges in the whole interval $(2, 2.5)$, for $0\leqslant q\leqslant 1$, with $D_0=2$ and $D_1=2.5$ $(D_q=D_1$ for $q>1)$.

Clearly, the very notion of standard scaling exponents can not adapt to P, due to the multiscaling concentrated at low q. However, in numerical work, P is usually analyzed by assuming a FSS form, e.g., on the basis of collapse fits. Thus, it is important to ask what should be the FSS form of distribution which most accurately reproduces the scaling of the moments of P. The optimal job in this respect is done by a distribution of the FSS form $s^{-\tau}p(s/L^D)$, whose qth moment exponent coincides with $\sigma_s(q)$ for $q>1$, and with the function represented by the straight dashed line in Fig. 1 for $q<1$. Of course, we put the fractal dimension equal to $D=D_1=2.5$ in p, because its opposite has to coincide with the asymptotic slope of σ_s. For such a scaling function one gets $\Sigma_s s^{1-\tau}p(s/L^D)\simeq L^{D(2-\tau)}$. Thus, $D(2-\tau)=2$ leads to $\tau=6/5$. Some accurate numerical determinations based on FSS [8] are very close to this τ, which remarkably coincides with that conjectured in Ref. [7]. The geometry of the various lines in Fig. 1 is very eloquent. In order to reduce the multiscaling of P to simple FSS, we have to shift to $s=0$ the contribution of $n=0$ avalanches in the histogram of P. This amounts to bend the actual σ_s curve in the interval $(0,1)$ into a straight segment (dashed), with intercept ζ on the vertical axis. Our results for D_q and τ give an indication of the problems arising when, e.g., one tries to enforce FSS collapses of data for the simultaneous determination of D and τ. Privileging collapse in the region of low s/L^D produces a simultaneous lowering of both D and τ, like when searching a solution of $D_q(2-\tau)=2$ for low values of q. The extreme case is $\tau=1$ with $D=D_0=2$. Emphasizing collapse at high s/L^D produces higher determinations of both D and τ. In this case a relatively poor sampling can lead to an overestimation of both exponents with respect to the above effective values. Indeed, we verified that σ_s and thus D_q tend to be overestimated systematically at large q.

The standard exponents of $n>0$ avalanches alone (ensemble 1) are τ_1 and D such that we can put $P_1(s,L)=s^{-\tau_1}p_1(s/L^D)$. From our results and from the constant gap -2.5 for σ_{1s}, it follows immediately $D=2.5$ and $\tau_1=1$.

Summarizing, we showed that standard FSS does not hold for the ASM in $2D$. Conservation of grains is guaranteed by

an intermittent mechanism, with rare, large avalanches producing the outflowing current. These avalanches are such to fully determine the moments of P from the first one upwards. The multiscaling spectrum of singularity strengths for P has peculiar features associated with the dominance of rare events. In particular $f(-5/2) = -1/2$ shows that rare avalanches have a fractal dimension equal to 5/2 and occur with frequency $\propto L^{-1/2}$. These features are such to give $\tau = 6/5$ and $D = 2.5$ as effective exponents representing the multiscaling within an imposed simple FSS framework. Not surprisingly, FSS based numerical determinations of τ and D are often quite close to the values mentioned above [8].

In spite of the fact that the approach of Ref. [7] does not take into account multiscaling features, the values of τ and D we propose coincide with those conjectured there. Not surprisingly, in view of our results, that conjecture has met problems of numerical verification [10]. A revisitation of wave, or, rather, cluster [16] properties in the light of the dominance of large avalanches and multiscaling can clarify the situation [17].

Crucial to the identification of our effective exponents are the constant gap $-(2+\zeta)$ for $q>1$ and the exact result $\sigma_s(1) = -2$ [2]. Thus, it is precisely $\zeta = 1/2$ that determines the conjectured τ and D in the present framework. The critical state of the ASM in $2D$ is expected to correspond to that of a non-unitary conformal field theory with central charge $c = -2$ [3]. The compatibility of multiscaling with other non-unitary conformal field theories has been pointed out recently [18]. It is also known [19] that in a theory with $c = -2$ a correlator of disorder operators possesses exactly the scaling dimension 1/2.

We expect intermittency and dominance of rare, large events to play a key role also in the physics of other SOC models. Indeed, $1D$ sandpiles most often display pronounced multiscaling features and intermittency [11–13]. Of course, the Abelian symmetry and the Laplacian characterization of the toppling dynamics made the analysis of these features relatively easy here, to the extent that exact properties of the model could be inferred. The case of ASM in higher dimensions, for which $\Sigma_{\mathbf{r}}\langle g(\mathbf{r},L)\rangle_L \propto L^2$ seems to hold irrespective of d [2,20], is also quite intriguing.

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